

Proof of universality of electrical conductivity at finite chemical potential

Sayan K. Chakrabarti¹, Shankhadeep Chakraborty², Sachin Jain³

*Institute of Physics,
Sachivalaya Marg, Bhubaneswar,
India-751 005.*

ABSTRACT

It was proposed in [1] that, for certain gauge theories with gravity duals, electrical conductivity at finite chemical potential is universal. Here we provide a general proof that, when matter stress tensor satisfies a compact constraint, electrical conductivity is universal. We further elaborate our result with several conformal as well as non-conformal gauge theories. We also discuss how boundary conductivity and universal conductivity of stretched horizon are related.

¹e-mail: sayan@iopb.res.in

²e-mail: sankha@iopb.res.in

³e-mail: sachjain@iopb.res.in

1 Introduction

The fluid/gravity correspondence provides us with two distinct fluids dual to a given black hole geometry: first, the fluid given by membrane paradigm, which is described by quantities at the black hole horizon and second, the fluid at the boundary of the space time known from gauge/gravity duality and is described by quantities at the boundary. By exploiting the fact that changing radial position in the bulk corresponds to RG flow in the boundary fluid, [2, 3] proposed a number of relations and even interpolation between them. For example, radial independence of certain quantities is used to show that, the shear viscosity (η) to entropy density (s) ratio ($\frac{\eta}{s}$) for both the fluid is same as well as the low frequency limit of electrical conductivities of these two distinct fluids computed at zero chemical potential, are related. However, the situation changes significantly at finite chemical potential in the boundary theory (which corresponds to charged black hole in the bulk), where radial independence exploited earlier in relating electrical conductivity of these two fluids, gets completely destroyed. One needs to solve flow equation in order to relate conductivities of these two fluids. Recently it was proposed in [1], based on few examples that, the boundary electrical conductivity is universal and that there exists a simple relation between the conductivities of the fluids at horizon and at boundary. It was further proposed that, at any radial position r , the conductivity is given by a simple expression which interpolates smoothly between the one computed at the horizon and at the boundary. In all the examples considered in the above mentioned work, the bulk theory was asymptotically AdS for which there exists a CFT on the boundary at finite temperature and chemical potential. Now, going beyond CFT, it would be interesting to see whether such universality holds good for other boundary theories such as non-conformal or Lifshitz like theories with finite chemical potentials. For gauge theories dual to charged Lifshitz like gravity backgrounds, it was shown in [1] that, the above mentioned universality does not hold. On the contrary, the electrical conductivity of other theories such as non-conformal fluid living on charged $D1$ brane [4] shows the same universality mentioned above (as shown later in the paper). So at present, there is no systematic way to answer which theories would show the universality in the electrical conductivity and which will not ². Rather than checking case by case, it is desirable to have a characterization for the theories which will show the proposed universality. In this paper, we show that only those gauge theories which have a gravity dual with particular matter content will show this universality. This further explains why charged Lifshitz like black holes do not show the universality.

The paper is structured as follows. Section 2 is a review of earlier works [1, 6, 7]. In this section we also discuss all the assumptions made in the gravity side. In section

²Note that a closed formula for the electrical conductivity at finite chemical potential has been presented for a wide class of models in [5]. Although the set up we will consider in this paper will be completely different from the one used in [5] which involves probe branes.

3, we find the condition in the gravity side under which the dual gauge theory will show the universality. Section 4 discusses several examples, which include theories at and away from conformality. This section also explains why the Lifshitz like theories do not show the universality. In section 5, we explicitly discuss on non-conformal gauge theory dual to charged Dp brane, and show the universality. Finally for completeness of this work we explicitly check the universality for cases with multiple charges. We further compute the thermal conductivity to viscosity ratio and show its universality. Finally we conclude the paper in section 6. In appendix A, we briefly discuss computation of electrical conductivity from gravity side and discuss the flow equations. In appendix B, we elaborate upon the condition that we get on energy momentum tensor.

2 What to prove?

This section is essentially a review of the earlier works. In this section we discuss the notations and assumptions in the gravity theory which are assumed to support gauge/gravity duality and write down the perturbation equation required for the computation of electrical conductivity (for details see [1]). Since we are interested in calculating the electrical conductivity in the presence of chemical potential, we consider the most general two-derivative gravity action of the following form

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} (R - \frac{1}{4g_{\text{eff}}^2(r)} F_{\mu\nu} F^{\mu\nu} + \text{Other terms}), \quad (2.1)$$

where $F^{\mu\nu}$ is the field-strength tensor of the $U(1)$ gauge field and $\frac{1}{g_{\text{eff}}^2}$ is the effective gauge coupling. The metric that we take is of the form

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r) \sum_{i=1}^{d-1} (dx^i)^2, \quad (2.2)$$

where r is the radial coordinate. We have assumed full rotational symmetry in x^i directions so that³ $g_{ij} = g_{xx}\delta_{ij}$, where i, j run over all the indices except r, t . We also assume that metric components depend on radial coordinate only. We shall work with the metric which has an event horizon⁴, where g_{tt} has a first order zero and g_{rr} has a first order pole. We also assume that all the other metric components are finite as well as non vanishing at the horizon. The boundary of the space time is at $r \rightarrow \infty$.

Since our aim is to compute the electrical conductivity using Kubo formula, it is sufficient to consider perturbations in the tensor (metric) and the vector (gauge fields)

³Let us note that, we are using the notation where $g_{\mu\nu}(r) \equiv g_{\mu\nu}$, $\frac{1}{g_{\text{eff}}^2(r)} \equiv \frac{1}{g_{\text{eff}}^2}$.

⁴For charged black holes, there exists inner horizons also.

modes around the black hole solution and keep other fields such as scalars unperturbed. We consider the perturbations of the form

$$g_{\mu\nu} = \mathbf{g}_{\mu\nu}^{(0)} + h_{\mu\nu} , \quad A_\mu = \mathbf{A}_\mu^{(0)} + \mathcal{A}_\mu , \quad (2.3)$$

where $\mathbf{g}_{\mu\nu}^{(0)}$ and $\mathbf{A}_\mu^{(0)}$ are background metric and gauge fields.

In order to determine electrical conductivity it is enough to consider perturbations in (tx^1) and (x^1x^1) components of the metric tensor and x^1 component of the gauge fields. Moreover one can choose the perturbations to depend on radial coordinate r , time t and one of the spatial coordinate say x^2 . A convenient ansatz with the above restrictions in mind is

$$h_{tx^1} = \mathbf{g}_{x^1x^1}^{(0)} T(r) e^{-i\omega t + iqx^2}, \quad h_{x^2x^1} = \mathbf{g}_{xx}^{(0)} Z(r) e^{-i\omega t + iqx^2}, \quad \mathcal{A}_{x^1} = \phi(r) e^{-i\omega t + iqx^2}. \quad (2.4)$$

Here ω and q represent the frequency and momentum in x^2 direction respectively and we set perturbations in the other components to be equal to zero. Our next step is to find linearized equations which follow from the equations of motion. It turns out that at the level of linearized equation and at zero momentum limit metric perturbation $Z(r)$ decouples from the rest. One can further eliminate $T(r)$ reducing it to equation for perturbations in gauge fields only. After substitution one finds the equation for perturbed gauge field to be

$$\frac{d}{dr}(N(r) \frac{d}{dr}\phi(r)) - \omega^2 N(r) g_{rr} g^{tt} \phi(r) + M(r) \phi(r) = 0, \quad (2.5)$$

with

$$N(r) = \sqrt{-g} \frac{1}{g_{\text{eff}}^2} g^{xx} g^{rr}, \quad (2.6)$$

and

$$M(r) = \left(\frac{1}{g_{\text{eff}}^2}\right)^2 \sqrt{-g} g^{xx} g^{rr} g^{tt} F_{rt} F_{rt}. \quad (2.7)$$

We can rewrite $M(r)$, in a better way as

$$M(r) = (2\kappa^2)^2 \rho^2 \frac{g_{rr} g_{tt}}{\sqrt{-g} g_{xx}}. \quad (2.8)$$

where,

$$\rho = \frac{1}{2\kappa^2 g_{\text{eff}}^2} \sqrt{-g} g^{rr} g^{tt} F_{rt}. \quad (2.9)$$

Let us note that the Maxwell equations can be written as,

$$\partial_\mu \left(\frac{1}{g_{\text{eff}}^2} \sqrt{-g} F^{\nu\mu} \right) = 0, \quad (2.10)$$

and we choose the gauge where only $A_t(r)$ component of the background gauge field is non zero (we work with electrically charged black hole).

For evaluating the conductivity in the low frequency limit and for non-extremal backgrounds, we only need to solve equations up to zeroth order in ω . To that order one finds,

$$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) + M(r)\phi(r) = 0. \quad (2.11)$$

The expression for electrical conductivity is given by (see [1, 6, 7] and Appendix A for details),

$$\begin{aligned} \sigma &= \frac{1}{2\kappa^2} \left(\sqrt{\frac{g_{rr}}{g_{tt}}} N(r) \right)_{r=r_h} \left(\frac{\phi(r_h)}{\phi(r \rightarrow \infty)} \right)^2 \\ &= \frac{1}{2\kappa^2} \left(\frac{1}{g_{\text{eff}}^2} g_{xx}^{\frac{d-3}{2}} \right)_{r=r_h} \left(\frac{\phi(r_h)}{\phi(r \rightarrow \infty)} \right)^2 \\ &= \sigma_H \left(\frac{\phi(r_h)}{\phi(r \rightarrow \infty)} \right)^2, \end{aligned} \quad (2.12)$$

where σ_H is the conductivity evaluated at the horizon and its expression is given by,

$$\sigma_H = \frac{1}{2\kappa^2 g_{\text{eff}}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}. \quad (2.13)$$

It was proposed in [1], that

$$\begin{aligned} \sigma &= \sigma_H \left(\frac{\phi(r_h)}{\phi(r \rightarrow \infty)} \right)^2 \\ &= \sigma_H \left(\frac{sT}{\epsilon + P} \right)^2, \end{aligned} \quad (2.14)$$

and

$$\frac{\phi(r)}{\phi(r_h)} = 1 + \frac{\rho}{sT} (A_t(r) - A_t(r_h)), \quad (2.15)$$

where Eq (2.15) at the boundary reduces to

$$\begin{aligned} \frac{\phi(r \rightarrow \infty)}{\phi(r_h)} &= 1 + \frac{\rho}{sT} \mu \\ &= \frac{\epsilon + P}{sT}. \end{aligned} \quad (2.16)$$

So in order to show Eq (2.14) we need to prove Eq (2.15). In the next section we show that Eq (2.15) indeed, is the solution to Eq (2.11).

3 Proof

In this section we want to prove Eq (2.14) or in other words, given a gravity background we want to understand under what conditions, the dual gauge theory will show behavior as in Eq (2.14). The way we shall proceed is, first we shall assume that the solution to Eq (2.11) is given by Eq (2.15). Then we shall use Einstein equation to find out the constraint that our assumption leads to and we obtain these constraints can be expressed in a compact form in terms of the stress energy momentum tensor of the matter content of the system.

We start by plugging Eq (2.15) in Eq (2.11). This gives,

$$\frac{d}{dr} \left(\sqrt{-g} \frac{1}{g_{\text{eff}}^2} g^{xx} g^{rr} \frac{\rho}{sT} \frac{d}{dr} A_t(r) \right) + (2\kappa^2)^2 \rho^2 \frac{g_{rr} g_{tt}}{\sqrt{-g} g_{xx}} \left(1 + \frac{\rho}{sT} (A_t(r) - A_t(r_h)) \right) = 0.$$

Using $F_{rt} = \frac{d}{dr} A_t$ and definition of charge density as in Eq (2.9) we obtain

$$\begin{aligned} 2\kappa^2 \frac{\rho^2}{sT} \frac{d}{dr} (g^{xx} g_{tt}) + (2\kappa^2)^2 \rho^2 \frac{g_{rr} g_{tt}}{\sqrt{-g} g_{xx}} \left(1 + \frac{\rho}{sT} (A_t(r) - A_t(r_h)) \right) &= 0, \\ \text{or, } \frac{1}{2\kappa^2} \frac{\sqrt{-g} g_{xx}}{g_{rr} g_{tt}} \frac{d}{dr} (g^{xx} g_{tt}) &= -sT \left(1 + \frac{\rho}{sT} (A_t(r) - A_t(r_h)) \right). \end{aligned} \quad (3.1)$$

Evaluating Eq (3.1) at $r = r_h$, we get

$$\left. \frac{1}{2\kappa^2} \frac{\sqrt{-g} g_{xx}}{g_{rr} g_{tt}} \frac{d}{dr} (g^{xx} g_{tt}) \right|_{r_h} = -sT. \quad (3.2)$$

Subtracting Eq (3.1) from Eq (3.2) we get

$$\begin{aligned} \frac{\sqrt{-g} g_{xx}}{g_{rr} g_{tt}} \frac{d}{dr} (g^{xx} g_{tt}) \Big|_{r_h}^r &= -2\kappa^2 \rho (A_t(r) - A_t(r_h)) \\ \Rightarrow \left[\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \frac{d}{dr} (g^{xx} g_{tt}) \right]_{r_h}^r &= -2\kappa^2 \rho A_t \Big|_{r_h}^r. \end{aligned} \quad (3.3)$$

Now we use Einstein equations to find out conditions under which Eq (3.3) is valid. Let us consider the background of the form given in Eq (2.2). The Einstein equation is given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= T_{\mu\nu}^{E.M.} + T_{\mu\nu}^{Matter} \\ &= \frac{1}{2g_{\text{eff}}^2} \left(F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) + T_{\mu\nu}^{Matter}, \end{aligned} \quad (3.4)$$

where $T_{\mu\nu}^{Matter}(r)$, will include all the other stuffs which may come from scalar fields, cosmological constant or any other fields present in the theory. Since only $A_t(r)$ is non-zero, we have $F_{rt} \neq 0$. Using Eq (3.4), we can write

$$R_t^t - \frac{1}{2}g_t^t R = \frac{1}{2g_{\text{eff}}^2} \left(F_{tr} F^{tr} - \frac{1}{4}g_t^t F_{\rho\sigma} F^{\rho\sigma} \right) + T_t^{t, Matter}, \quad (3.5)$$

$$R_x^x - \frac{1}{2}g_x^x R = -\frac{1}{2g_{\text{eff}}^2} \frac{1}{4}g_x^x F_{\rho\sigma} F^{\rho\sigma} + T_x^{x, Matter}. \quad (3.6)$$

After subtracting Eq (3.5) from Eq (3.6), we get

$$\sqrt{-g}R_t^t - \sqrt{-g}R_x^x = \frac{1}{2g_{\text{eff}}^2} \sqrt{-g} F^{rt} F_{rt} + \sqrt{-g}(T_t^{t, Matter}(r) - T_x^{x, Matter}(r)). \quad (3.7)$$

For the metric of the form in Eq (2.2), following relations hold

$$\sqrt{-g}R_t^t = -\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-1}{2}} \frac{d}{dr} g_{tt}}{2g_{rr}^{\frac{1}{2}} g_{tt}^{\frac{1}{2}}} \right), \quad (3.8)$$

$$\sqrt{-g}R_x^x = -\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-3}{2}} g_{tt}^{\frac{1}{2}} \frac{d}{dr} g_{xx}}{2g_{rr}^{\frac{1}{2}}} \right), \quad (3.9)$$

which, after substituting in Eq (3.7), we get,

$$-\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-1}{2}} \frac{d}{dr} g_{tt}}{2g_{rr}^{\frac{1}{2}} g_{tt}^{\frac{1}{2}}} \right) + \frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-3}{2}} g_{tt}^{\frac{1}{2}} \frac{d}{dr} g_{xx}}{2g_{rr}^{\frac{1}{2}}} \right) = \frac{1}{2g_{\text{eff}}^2} \sqrt{-g} F^{rt} F_{rt} + \sqrt{-g}(T_t^{t, Matter} - T_x^{x, Matter}). \quad (3.10)$$

Upon further simplification, this reduces to

$$-\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d+1}{2}} \frac{d}{dr} (g^{xx} g_{tt})}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \right) = 2\kappa^2 \rho \frac{d}{dr} A_t + 2\sqrt{-g}(T_t^{t, Matter}(r) - T_x^{x, Matter}(r)). \quad (3.11)$$

Integrating above equation we get

$$\left(\frac{g_{xx}^{\frac{d+1}{2}} \frac{d}{dr} (g^{xx} g_{tt})}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \right) \Bigg|_{r_h}^r = -2\kappa^2 \rho A_t \Bigg|_{r_h}^r + 2 \int_{r_h}^r dr \sqrt{-g}(T_t^{t, Matter}(r) - T_x^{x, Matter}(r)). \quad (3.12)$$

Thus, if we impose the condition that

$$T_t^{t, Matter}(r) = T_x^{x, Matter}(r), \quad (3.13)$$

then we get

$$\left(\frac{g_{xx}^{\frac{d+1}{2}} \frac{d}{dr} (g^{xx} g_{tt})}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \right) \Bigg|_{r_h}^r = -2\kappa^2 \rho A_t \Bigg|_{r_h}^r, \quad (3.14)$$

which⁵ is same as Eq (3.3). Hence, what we have shown is, if the gravity background satisfies Eq (3.13), then the dual gauge theory will satisfy Eq (2.12). We suspect that whenever the boundary theory is in the Minkowski space, the condition imposed by Eq (3.13) on the stress-energy tensor (barring the electromagnetic part) will hold true. This was also observed in [8, 9] in the context of proving the universality of shear viscosity. In the following section, we elaborate upon the above condition considering several examples.

4 Examples

In all of our examples in this section we will take the metric, gauge fields and other form fields as the functions of coordinate r only. It was observed in [8, 9] that if the scalar and other form fields are functions of the coordinate r only and if the boundary theory lives on the Minkowski space, then $T_{\mu\nu}^{\text{Matter}} \sim g_{\mu\nu}(\dots)$, which in turn implies the condition given by Eq (3.13). In what follows, in this section, we first discuss the boundary theories which live on Minkowski space-time where we will find explicitly that the Eq (3.13) holds good. Next, we discuss one example where the boundary theory does not live on the Minkowski space-time, namely the asymptotically Lifshitz like space-time, where the condition does not hold.

- **Boundary theories living on Minkowski space-time**

- **Conformal boundary theories:** Let us note that Reissner Nordström and R-charged black holes in various dimensions in asymptotically AdS space (as already checked in [1]) as well as any other background which satisfies Eq (3.13), should satisfy Eq (2.12).
- **Non-conformal boundary theory:** Non-conformal theories such as gauge theory dual to charged Dp brane satisfies Eq (2.12). We shall check this explicitly in the next section.

- **Boundary theory dual to charged Lifshitz like black hole:** For this case it was computed in [1] that

$$\sigma_B \neq \sigma_H \left(\frac{sT}{\epsilon + P} \right)^2. \quad (4.1)$$

⁵For the backgrounds which satisfies Eq (3.13), it is interesting to note that, if we set $r \rightarrow \infty$, and use first law of thermodynamics as well as the fact that $sT_H = \frac{1}{2\kappa^2} \left(\frac{\frac{d+1}{2}}{\frac{1}{2} \frac{g_{xx}}{g_{tt}^2 g_{rr}^2}} \frac{d}{dr} (g^{xx} g_{tt}) \right) \Big|_{r_h}$, we have

$\epsilon + P = \frac{1}{2\kappa^2} \left(\frac{\frac{d+1}{2}}{\frac{1}{2} \frac{g_{xx}}{g_{tt}^2 g_{rr}^2}} \frac{d}{dr} (g^{xx} g_{tt}) \right) \Big|_{r \rightarrow \infty}$ from Eq (3.3). Let us note that we should add the Gibbons-Hawking term and counter terms (see [10]) in order to get finite values.

Now the above result can be understood easily. Let us consider the following action in $(d+2)$ -dimensional space time (see for details in [11, 12])

$$S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} (R - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{2}m^2 A^2 - \frac{1}{4}F_1^2). \quad (4.2)$$

The corresponding equations of motion are given as follows,

$$\begin{aligned} \partial_\mu(\sqrt{-g}F^{\mu\nu}) &= m^2\sqrt{-g}A^\nu, \quad \partial_\mu(\sqrt{-g}F_1^{\mu\nu}) = 0, \\ R_{\mu\nu} &= \frac{2}{d}\Lambda g_{\mu\nu} + \frac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{2}F_{1,\mu\lambda}F_{1,\nu}{}^\lambda + \frac{1}{2}m^2 A_\mu A_\nu \\ &\quad - \frac{1}{4d}F^2 g_{\mu\nu} - \frac{1}{4d}F_1^2 g_{\mu\nu}. \end{aligned} \quad (4.3)$$

From the above equation we can find the energy momentum tensor. Let us write it in the form $T_{\mu\nu}^{total} = T_{\mu\nu}^{E.M.} + T_{\mu\nu}^{Matter}$, where $T_{\mu\nu}^{E.M.}$ contains contribution from gauge field $F_{1,\mu\nu}$ whereas other fields contributes to $T_{\mu\nu}^{Matter}$. Let us note that the massive gauge field A_μ ; was introduced to get the Lifshitz like scaling. If we take only non-vanishing components of gauge field to be A_t , then it is easy to see that

$$\begin{aligned} T_t^{t,Matter} - T_x^{x,Matter} &= \frac{1}{2}F_{tr}F^{tr} + \frac{1}{2}m^2 A_t A^t \\ &\neq 0, \end{aligned} \quad (4.4)$$

where $F_{rt} = \frac{d}{dr}A_t$ and also note that $g_t^t = g_x^x = 1$. This provides us with the explanation of Eq (4.1).

5 Away from conformality:

We shall motivate the purpose of this section by giving example of single charged $D1$ brane case recently considered in [4]. Then we shall check explicitly the validity of Eq (2.12) for non-conformal gauge theories dual to general charged Dp brane as well as give general results for multiple charged Dp brane.

- **Electrical conductivity for charged $D1$ brane:** Let us consider the following action

$$\begin{aligned} I &= \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left[R(g) - \frac{8}{9} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \Psi^2 e^{-\frac{4}{3}\phi} F_{\mu\nu} F^{\mu\nu} \right. \\ &\quad \left. - \frac{1}{2\Psi^2} \partial_\mu \Psi \partial^\mu \Psi + \frac{2}{3\Psi} \partial_\mu \phi \partial^\mu \Psi + \frac{12}{L^2} e^{\frac{4}{3}\phi} (1 + \Psi^{-1}) \right]. \end{aligned} \quad (5.1)$$

Here again one can easily check that Eq (3.13) is satisfied, so here our general formula Eq (2.12) must hold. In the following we shall check this explicitly.

The metric, gauge field and scalar fields are given by

$$\begin{aligned} ds^2 &= (-c_T^2 dt^2 + c_X^2 dz^2 + c_R^2 dr^2), \\ c_T^2 &= \left(\frac{r}{L}\right)^8 K, \quad c_X^2 = \left(\frac{r}{L}\right)^8 H, \quad c_R^2 = \frac{H}{K} \left(\frac{r}{L}\right)^2, \\ A_t &= -\frac{r_0^3 l}{L^2(r^2 + l^2)}, \quad \phi = -3 \log\left(\frac{r}{L}\right), \quad \Psi = 1 + \frac{l^2}{r^2}. \end{aligned} \quad (5.2)$$

Here H and K are defined as

$$H = 1 + \frac{l^2}{r^2}, \quad K = 1 + \frac{l^2}{r^2} - \frac{r_0^6}{r^6}. \quad (5.3)$$

Different thermodynamic quantities are given by,

$$T = \frac{1}{2\pi L^3} \frac{r_H^5}{r_0^3} (3 + 2k), \quad s = \frac{1}{4G_3} \frac{r_0^3 r_H}{L^4}, \quad (5.4)$$

where k is given by

$$k = \frac{l^2}{r_H^2}, \quad (5.5)$$

and r_H is the radius of the horizon which is given by the largest root of the equation

$$r_H^6 + r_H^4 l^2 - r_0^6 = 0. \quad (5.6)$$

The energy density (ϵ) and the pressure (p) is given by

$$\epsilon = \frac{1}{4\pi G_3} \frac{r_0^6}{L^7}, \quad p = \frac{1}{8\pi G_3} \frac{r_0^6}{L^7} = \frac{\epsilon}{2}. \quad (5.7)$$

The charge density ρ and its conjugate the chemical potential μ are given by

$$\rho = \frac{1}{8\pi G_3} \frac{r_0^3 l}{L^5}, \quad \mu = A_t(r)|_{r \rightarrow \infty} - A_t(r)|_{r_H} = \frac{l r_H^4}{L^2 r_0^3}. \quad (5.8)$$

So conductivity should be,

$$\begin{aligned} \sigma &= \frac{1}{16\pi G_3} \frac{1}{g_{\text{eff}}^2} g_{xx}^{-\frac{1}{2}} \Big|_{r=r_h} \left(\frac{sT}{\epsilon + P} \right)^2 \\ &= \frac{1}{16\pi G_3} \Psi^2 e^{-\frac{4}{3}\phi} g_{xx}^{-\frac{1}{2}} \Big|_{r=r_h} \left(\frac{sT}{\epsilon + P} \right)^2 \\ &= \frac{1}{16\pi G_3} \frac{(2k + 3)^2}{9\sqrt{1 + k}}, \end{aligned} \quad (5.9)$$

which is same as the one computed in [4]. In that paper authors also computed electrical conductivity for four equal charge case. The results follow from Eq (2.12) in a straight forward manner.

- **Thermal conductivity :** In [4], thermal conductivity to bulk viscosity (ζ) ratio for both single charge and equal four charge case was computed to be

$$\frac{\kappa_T}{\zeta T} \mu^2 = 4\pi^2 L^2. \quad (5.10)$$

- Recently it was noted in [1, 13], that this particular ratio remains unchanged even if we take multiple chemical potentials or set chemical potential to zero. In the following we check this again.

5.1 Uncharged Dp brane:

Let us consider the following background

$$ds^2 = -g^{\frac{n+1}{d-2}} r^{\frac{n+1}{d-2}} f(r) dt^2 + g^{\frac{n+1}{d-2}} r^{\frac{n+1}{d-2}} \sum_{i=1}^p dx_i^2 + g^{1-n+\frac{n+1}{d-2}} r^{1-n+\frac{n+1}{d-2}} \frac{1}{f(r)} dr^2, \quad (5.11)$$

where $f(r) = 1 - \frac{2m}{r^{n-1}}$, $d = p + 2$, $n = 10 - d$ and g is a constant ($= \frac{1}{L}$, in the notation used in [4]). Note that this background is exactly same as the one considered in [14]. We have written it differently. As it was argued in [14], in order to determine electrical conductivity one should consider the following Maxwell equation,

$$S_{Maxwell} = -\frac{1}{16\pi G} \int dx^{p+2} \sqrt{-g_{p+2}} \frac{1}{4 g_{eff}^2} F_{\mu\nu} F^{\mu\nu}, \quad (5.12)$$

where

$$g_{eff}^2 = (gr)^{-\frac{a^2(D-2)}{2(d-2)}}, \quad a^2 = 4 - 2 \frac{(n-1)(D-n-1)}{D-2}, \quad (5.13)$$

and D is the higher dimensional space from where we are reducing the metric down to d dimensions (for our case $D = 10$). Various thermodynamic quantities are given by,

$$T = \frac{n-1}{4\pi r_h} (gr_h)^{\frac{n-1}{2}}, \quad s = \frac{1}{4G} g_{xx}^{\frac{p-2}{2}} \Big|_{r_H}, \quad (5.14)$$

and $\epsilon + P = sT$. The electrical conductivity is given by

$$\begin{aligned} \sigma &= \frac{1}{16\pi G} \frac{1}{g_{eff}^2} g_{xx}^{\frac{p-2}{2}} \Big|_{r_H} \\ &= \frac{1}{16\pi G} (gr_h)^{\frac{7-n}{2}}. \end{aligned} \quad (5.15)$$

It is easy to see that,

$$\begin{aligned} D_R &= \frac{\sigma}{\chi} \\ &= \frac{7-p}{8\pi T}, \end{aligned} \quad (5.16)$$

as was shown in [14], where

$$\begin{aligned}\chi &= \frac{\rho}{\mu} \\ &= \frac{1}{8\pi G} g^3 r_h^2.\end{aligned}\tag{5.17}$$

Let us note that, though ρ and μ go to zero separately for uncharged Dp brane, χ in Eq (5.17), remains non-zero. Now using expression for thermal conductivity, $\kappa_T = \frac{(\epsilon+p)^2 \sigma}{\rho^2 T}$, we get

$$\begin{aligned}\frac{\kappa_T}{\eta T} \mu^2 &= 4\pi \left(\frac{\sigma}{\chi} \right)^2 \frac{s}{\sigma} \\ &= \frac{4\pi^2}{g^2}.\end{aligned}\tag{5.18}$$

Note that, from Eq (5.18), we see that thermal conductivity to viscosity ratio is same for any uncharged Dp brane. Also note, to match with charged $D1$ brane, replace η by bulk viscosity and $g = \frac{1}{L}$.

Our next aim is to see whether for charged non-conformal theories dual to charged Dp brane, thermal conductivity to viscosity ratio remains $\frac{4\pi^2}{g^2}$.

5.2 Charged Dp brane

Let us consider the background obtained from Kaluza-Klein spherical reduction of single charged rotating black Dp brane to d dimension (see for details [15–17]).

$$ds^2 = -g^{\frac{n+1}{d-2}} r^{\frac{n+1}{d-2}} h^{-\frac{d-3}{d-2}} f(r) dt^2 + g^{\frac{n+1}{d-2}} r^{\frac{n+1}{d-2}} h^{\frac{1}{d-2}} \sum_{i=1}^p dx_i^2 + g^{1-n+\frac{n+1}{d-2}} r^{1-n+\frac{n+1}{d-2}} h^{\frac{1}{d-2}} \frac{1}{f(r)} dr^2,\tag{5.19}$$

where

$$f(r) = h - \frac{2m}{r^{n-1}}, \quad h = \prod_{i=1}^b (1 + H_i), \quad H_i = 1 + \frac{l_i^2}{r^2},\tag{5.20}$$

where b is the number of independent gauge fields (which is same as number of independent spins that a higher dimensional Dp brane can have before compactification). The action is of the form

$$S = \frac{1}{16\pi G} \int d^{p+2}x \sqrt{-g} \left[R - \frac{1}{4} \sum_{i=1}^b \frac{1}{X_i^2} F_{\mu\nu}^i F^{i\ \mu\nu} + \text{all the other terms..} \right],\tag{5.21}$$

where

$$X_i = g^{-\frac{a^2(D-2)}{4(d-2)}} r^{-\frac{a^2(D-2)}{4(d-2)}} h^{\frac{d-3}{2(d-2)}} \frac{1}{H_i},\tag{5.22}$$

and

$$A_t^i = -\sqrt{2mg}^{\frac{n-3}{2}} \frac{1 - \frac{1}{H_i}}{l_i}. \quad (5.23)$$

In the following we define all the required thermodynamic quantities. The expression for charge density is,

$$\rho_i = \frac{1}{8\pi G} \sqrt{2mg}^{\frac{n+3}{2}} l_i, \quad (5.24)$$

the chemical potentials are given by

$$\mu_i = \sqrt{2mg}^{\frac{n-3}{2}} \frac{l_i}{r_h^2 H_i(r_h)}. \quad (5.25)$$

The Hawking temperature is given by

$$T = \frac{\sqrt{m}}{\sqrt{2\pi} r_h} g^{\frac{n-1}{2}} \left(\frac{n-1}{2} - \frac{1}{r_h^2} \sum_{j=1}^b \frac{l_j^2}{H_j(r_h)} \right). \quad (5.26)$$

The expression for entropy and other required quantities are

$$s = \frac{1}{4G} g^{\frac{n+1}{2}} r_h \sqrt{2m}, \quad \epsilon + P = \frac{(n-1)m}{8\pi G} g^n. \quad (5.27)$$

The equation⁶ that we have to solve in order to find out conductivity is given by

$$\frac{d}{dr} \left(N_i \frac{d}{dr} \phi_i(r) \right) + \sum_{j=1}^m M_{ij} \phi_j(r) = 0. \quad (5.28)$$

where

$$N_i = \sqrt{-g} \frac{1}{X_i^2} g^{xx} g^{rr}, \quad (5.29)$$

and

$$M_{ij} = F_{rt}^i \sqrt{-g} \frac{1}{X_i^2} g^{xx} g^{rr} g^{tt} \frac{1}{X_j^2} F_{rt}^j. \quad (5.30)$$

Plugging the background values we can show

$$N_i = g^3 r^3 f(r) H_i^2 \frac{1}{h}, \quad M_{ij} = -8m l_i l_j g^3 r^{-n} \frac{1}{h}. \quad (5.31)$$

• **Single charge case:** Here we have

$$\sigma = \frac{1}{16\pi G} \frac{1}{X^2} g_{xx}^{\frac{p-2}{2}} \Big|_{r_H} \left(\frac{sT}{\epsilon + P} \right)^2. \quad (5.32)$$

Next using the fact that,

$$\frac{\rho}{\mu} = \frac{1}{8\pi G} g^3 r_h^2 H(r_h), \quad (5.33)$$

⁶Unless explicitly mentioned, there is no sum over repeated indices i, j .

we get

$$\frac{K_T \mu^2}{\eta T} = \frac{4\pi^2}{g^2}, \quad (5.34)$$

which is same as we get for uncharged case.

- **Multicharge case:** For multicharge case, there is an analog of Eq (2.14). As it was proposed in [1], that for multicharge case

$$\rho_i \sigma_{ij}^{-1} \rho_j = \rho_i \sigma_{H,ii}^{-1} \rho_i \left(\frac{\epsilon + P}{sT} \right)^2, \quad (5.35)$$

where $\sigma_{H,ii}^{-1}$ is the inverse of electrical conductivity evaluated at the horizon and only depends on geometrical quantities evaluated at the horizon. The expression for electrical conductivity at the horizon is given by,

$$\begin{aligned} \sigma_{H,ii} &= \frac{1}{16\pi G} G_{ii}(r) g_{xx}^{\frac{p-2}{2}} \Big|_{r=r_h} \\ &= \frac{1}{16\pi G} \frac{1}{X_i^2} g_{xx}^{\frac{p-2}{2}} \Big|_{r=r_h} \\ &= \frac{g^{\frac{7-n}{2}} r_h^3 H_i^2(r_h)}{16\sqrt{2m} \pi G}. \end{aligned} \quad (5.36)$$

Using this result, it can be easily shown that,

$$\frac{K_T \sum_{i=1}^b \mu_i^2}{\eta T} = \frac{4\pi^2}{g^2}. \quad (5.37)$$

For $D1$ brane η is replaced by $\frac{s}{4\pi}$ (which is same as bulk viscosity for single charge case or equally charged $D1$ brane case as shown in [4]).

- **D1 brane with four unequal charges:** In this case, the coupled set of equations for i^{th} field are given by

$$\frac{d}{dr} \left(N_i \frac{d}{dr} \phi_i(r) \right) + \sum_{j=1}^4 M_{ij} \phi_j(r) = 0, \quad (5.38)$$

where index i , can take value from 1 to 4 (there is no sum over i in the above) and

$$N_i = g^3 r^3 f(r) H_i^2 \frac{1}{h}, \quad M_{ij} = -8m l_i l_j g^3 r^{-7} \frac{1}{h}, \quad h = \prod_{i=1}^4 (1 + H_i). \quad (5.39)$$

Demanding regularity (ingoing boundary condition) at the horizon and at the boundary $\phi_i = \phi_i^0$, we get the solution to 4 coupled equation to be

$$\phi_i = \frac{\phi_i^0 + \frac{l_i}{6r^2} (6l_i \phi_i^0 - 2 \sum_{j=1}^4 \phi_j^0 l_j)}{H_i^2}. \quad (5.40)$$

We can now compute the conductivity by using the expression discussed in reference [1]. The expression for diagonal part of electrical conductivity is given by

$$\sigma_{ii} = \frac{9r_h^4 + 12r_h^2 l_i^2 + 3l_i^4 + l_i^2 \sum_{j=1}^4 l_j^2}{144\sqrt{2m} \pi G r_h}, \quad (5.41)$$

whereas off diagonal part of the conductivity is given by

$$\sigma_{ij} = -\frac{l_i l_j}{144\sqrt{2m} \pi G r_h} (6r_h^2 + \sum_{k=1}^4 l_k^2 - 3(l_i^2 + l_j^2)). \quad (5.42)$$

We can now explicitly check that, for multicharge case

$$\rho_i \sigma_{ij}^{-1} \rho_j = \rho_i \sigma_{H,ii}^{-1} \rho_i \left(\frac{\epsilon + P}{sT} \right)^2, \quad (5.43)$$

where

$$\sigma_{H,ii} = \frac{r_h^3 H_i^2(r_h)}{16\sqrt{2m} \pi G}, \quad (5.44)$$

is the electrical conductivity evaluated at the horizon and depends only on the geometrical quantities evaluated at the horizon. The thermal conductivity (K_T) can be computed using Eq (5.35) and Eq (5.36), and

$$K_T = \left(\frac{\epsilon + P}{T} \right)^2 \frac{T}{\sum_{i,j=1}^4 \rho_i \sigma_{ij}^{-1} \rho_j}. \quad (5.45)$$

Plugging all the expressions we get,

$$K_T = \frac{\pi}{g^2} \frac{sT}{\sum_{i=1}^4 \mu_i^2}. \quad (5.46)$$

We now compute the required ratio

$$\frac{K_T \sum_{i=1}^4 \mu_i^2}{\frac{s}{4\pi} T} = \frac{4\pi^2}{g^2}. \quad (5.47)$$

6 Conclusion

In this paper we have shown that, for $\mu \neq 0$, given that the form of Maxwell part of the action is

$$S = - \int d^{d+1}x \sqrt{-g} \frac{1}{4g_{eff}^2} F_{MN} F^{MN}, \quad (6.1)$$

the electrical conductivity at the boundary is given by

$$\begin{aligned}\sigma_B &= \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT)^2}{(\epsilon + P)^2} \\ &= \sigma_H \frac{(sT)^2}{(\epsilon + P)^2},\end{aligned}\tag{6.2}$$

where $\sigma_H = \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}$, is the electrical conductivity evaluated radially at the horizon. Following [1], we can argue that once the real part of the conductivity is known, the imaginary part of conductivity is automatically fixed. To summarize, in the presence of chemical potential the electrical conductivity can be expressed as

$$\lambda = -\frac{i}{\omega} \left(\frac{g_{tt}}{g_{xx}} \right)_{r \rightarrow \infty} \frac{\rho^2}{\epsilon + P} + \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT)^2}{(\epsilon + P)^2}.\tag{6.3}$$

Let us mention here that the imaginary part of the conductivity has a pole at $\omega \rightarrow 0$ limit because of the translational invariance of the system. If one uses the Krammers-Kronig relation

$$\Im(\lambda(\omega)) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Re(\lambda(\omega'))}{\omega' - \omega} d\omega',\tag{6.4}$$

then one can find that the real part of the conductivity contains a delta function iff the imaginary part has a pole. As we have found a pole in the imaginary part of the conductivity, it follows that real part has a delta function singularity at $\omega = 0$. So, strictly speaking DC conductivity that we have computed is low frequency limit of AC conductivity or more precisely expression for conductivity is valid for $\omega \rightarrow 0^+$, see [4, 18] for a nice discussion.

It is interesting to note that, following [1], the cutoff dependent conductivity can be computed which interpolates smoothly between the results at the horizon and at the boundary. At any cutoff r_c the expression for electrical conductivity⁷ can be written as

$$\lambda = -\frac{i}{\omega} \left(\frac{g_{tt}}{g_{xx}} \right)_{r_c} \left(\frac{\rho^2}{\epsilon + P} \right)_{r \rightarrow \infty} + \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT)^2}{(\epsilon + P)^2} \Big|_{r=r_c},\tag{6.5}$$

where $r \rightarrow \infty$ is the boundary of the space time. It is interesting to compare our results with the results obtained from the membrane paradigm arguments. We have seen, that

⁷Let us note that, at any radius r_c , the local temperature and the chemical potential can be given by $T_c = \frac{T_H}{\sqrt{g_{tt}(r_c)}}$ and $\mu_c = \frac{A_t(r_c) - A_t(r_h)}{\sqrt{g_{tt}(r_c)}}$ respectively. Assuming first law of thermodynamics $\epsilon(r_c) + P(r_c) = sT_c + \rho\mu_c$ to hold at any radius and using Eq (2.15) we get

$$\frac{\phi(r_c)}{\phi(r_h)} = \frac{sT}{\epsilon + P} \Big|_{r=r_c},$$

and consequently Eq (6.5).

irrespective of the theory, the horizon conductivity is given by

$$\sigma_H = \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}, \quad (6.6)$$

whereas the universal conductivity of the membrane is given by

$$\sigma_{membrane} = \frac{1}{g_{eff}^2} \Big|_{r=r_h}. \quad (6.7)$$

So we conclude that the horizon conductivity is given by,

$$\sigma_H = \sigma_{mem} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}. \quad (6.8)$$

We have also seen that for the background as of the form Eq (2.2), if Eq (3.13) is satisfied then the boundary conductivity can be related to horizon conductivity using thermodynamic quantities. More precisely we can write,

$$\begin{aligned} \sigma_B &= \sigma_H \frac{(sT)^2}{(\epsilon + P)^2} \\ &= \sigma_{mem} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT)^2}{(\epsilon + P)^2}. \end{aligned} \quad (6.9)$$

Since mass dimension of electrical conductivity is $d - 3$, one can understand the factor $g_{xx}^{\frac{d-3}{2}}$ as the converter of the length scale of the boundary to the proper length at the horizon [2, 6]. It would be very interesting to understand the meaning of extra factor $(\frac{sT}{\epsilon+P})^2$ that appears in the formula due to presence of chemical potential. At this moment it is not quite clear to us how to interpret it directly from the constraint Eq (3.13) which appears to be related to Lorentz invariance of the vacuum of the field theory. Let us note that, at zero chemical potential

$$\begin{aligned} \sigma_B &= \sigma_H \\ &= \sigma_{mem} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}, \end{aligned} \quad (6.10)$$

as was shown in [2].

In our result of electrical conductivity, σ_H is given entirely in terms of gravity theory. A natural question that arises, whether it is possible to give an intrinsic meaning to the expression of conductivity in terms of field theory quantities? This will put the formula for electrical conductivity in the same footing as celebrated universal result for $\frac{\eta}{s}$. Answer to this comes from the expression of thermal conductivity to viscosity ratio. As it was shown in [7], electrical conductivity can be expressed in terms of the field theory quantities alone.

We have seen that, using universality of electrical conductivity, another universality of thermal conductivity to shear viscosity ratio might be shown easily. We leave the proof of the universality of thermal conductivity to viscosity as a scope for the future work.

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A Expression for conductivity and flow equation

The electrical conductivity is usually computed from current-current correlator

$$\begin{aligned}\lambda &= -\lim_{\omega \rightarrow 0} \frac{G_{xx}(\omega, q=0)}{i\omega} \\ &= \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int_{-\infty}^{\infty} dt e^{-i\omega t} \int d\vec{x} \langle [J_x(t, \vec{x}), J_x(0, \vec{0})] \rangle.\end{aligned}\quad (\text{A.1})$$

The current-current correlator can be computed by taking second derivative of effective action which reproduces the Eq.(2.11) with respect to boundary fields. Let us note that at low frequency, the electrical conductivity (see [1] for details), at any radius r can be written as

$$\begin{aligned}\lambda(r) &= -\frac{1}{\omega} \left(\frac{g_{tt}}{g_{xx}} \right)_r \left(\frac{\rho^2}{\epsilon + P} \right)_{r \rightarrow \infty} + \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT)^2}{(\epsilon + P)^2} \Big|_{r=r_c} \\ &= -\frac{1}{\omega} \left(\frac{g_{tt}}{g_{xx}} \right)_r \left(\frac{\rho^2}{\epsilon + P} \right)_{r \rightarrow \infty} + \frac{1}{g_{eff}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h} \frac{(sT_H)^2}{(sT_H + \rho(A_t(r) - A_t(r_h)))^2}.\end{aligned}\quad (\text{A.2})$$

We also know how to relate the boundary transport coefficient with the horizon transport coefficient (see Eq (6.9)). So one of the remaining motivations to study flow equation can be to compute electrical conductivity away from low frequency limit (see [19]). Effective action which will reproduce Eq (2.5) can be written as

$$\begin{aligned}S &= \frac{1}{2\kappa^2} \int \frac{d^d q}{(2\pi)^d} dr \left[-\frac{1}{2} N(r) \frac{d}{dr} \phi(r, \omega) \frac{d}{dr} \phi(r, -\omega) \right. \\ &\quad \left. + \frac{1}{2} M(r) \phi(r, \omega) \phi(r, -\omega) - \omega^2 \frac{1}{2} N(r) g_{rr} g^{tt} \phi(r, \omega) \phi(r, -\omega) \right],\end{aligned}\quad (\text{A.3})$$

where

$$N(r) = \sqrt{-g} \frac{1}{g_{eff}^2} g^{xx} g^{rr} \quad (\text{A.4})$$

and

$$M(r) = \left(\frac{1}{g_{eff}^2} \right)^2 \sqrt{-g} g^{xx} g^{rr} g^{tt} F_{rt} F_{rt}. \quad (\text{A.5})$$

The canonical momentum ⁸ conjugate to field $\phi(r, \omega)$ can be written as,

$$\begin{aligned}\Pi(r, \omega) &= \frac{\delta S}{\delta \phi'(r, \omega)} \\ &= -\frac{1}{2\kappa^2} N(r) \phi',\end{aligned}\tag{A.6}$$

where $\phi' = \frac{d}{dr}\phi$. Now we can define electrical conductivity as

$$\sigma(r, \omega) = \frac{\Pi(r, \omega)}{i\omega\phi(r)}.\tag{A.7}$$

If we define $\sigma(r, \omega) = i \Im(\sigma(r, \omega)) + \Re(\sigma(r, \omega))$, then we get

$$\begin{aligned}\Re(\sigma(r, \omega)) &= \Re\left(\frac{\Pi(r, \omega)}{i\omega\phi(r)}\right) \\ &= \Re\left(\frac{\Pi(r, \omega)\phi(r)}{i\omega\phi^2(r)}\right) \\ &= -\Im\left(\frac{\Pi(r, \omega)\phi(r)}{\omega\phi^2(r)}\right).\end{aligned}\tag{A.8}$$

Using the fact that

$$\frac{d}{dr}\Im[\Pi(r, \omega)\phi(r)] = 0,\tag{A.9}$$

and

$$\lim_{r \rightarrow r_h} \frac{d}{dr}\phi(r) = -i\omega \lim_{r \rightarrow r_h} \sqrt{\frac{g_{rr}}{g_{tt}}}\phi(r) + \mathcal{O}(\omega^2).\tag{A.10}$$

we get (in the limit $\omega \rightarrow 0$)

$$\begin{aligned}\Re(\sigma(r)) &= \frac{1}{2\kappa^2} \left(\sqrt{\frac{g_{rr}}{g_{tt}}} N(r) \right)_{r=r_h} \left(\frac{\phi(r_h)}{\phi(r)} \right)^2 \\ &= \frac{1}{2\kappa^2} \left(\frac{1}{g_{\text{eff}}^2} g_{xx}^{\frac{d-3}{2}} \right)_{r=r_h} \left(\frac{\phi(r_h)}{\phi(r)} \right)^2 \\ &= \sigma_H \left(\frac{\phi(r_h)}{\phi(r)} \right)^2,\end{aligned}\tag{A.11}$$

where σ_H is the conductivity evaluated at the horizon and its expression is given by,

$$\sigma_H = \frac{1}{2\kappa^2 g_{\text{eff}}^2} g_{xx}^{\frac{d-3}{2}} \Big|_{r=r_h}.\tag{A.12}$$

⁸Since $\phi = A_x$, momentum $\Pi \equiv J^x$, where J^x is the current corresponding to A_x fluctuation.

At the boundary we get

$$\Re(\sigma(r \rightarrow \infty)) = \sigma_H \left(\frac{\phi(r_h)}{\phi(r \rightarrow \infty)} \right)^2. \quad (\text{A.13})$$

The imaginary part of conductivity can be written as

$$\Im(\sigma(r \rightarrow \infty)) = -\frac{1}{\omega} \lim_{r \rightarrow \infty} \lim_{\omega \rightarrow 0} \frac{\Pi(r, \omega)}{\phi(r, \omega)} \quad (\text{A.14})$$

We refer reader to [1] for details and regarding how to compute electrical conductivity for multiple charge case. In the following we turn our attention to the case away from low frequency limit. Taking derivative with respect to r , we get

$$\frac{d}{dr} \sigma(r, \omega) = i\omega \frac{2\kappa^2}{N(r)} \left[\sigma^2(r, \omega) + \left(\frac{1}{2\kappa^2} \right)^2 (N^2(r) g_{rr} g^{tt} - \frac{1}{\omega^2} M(r) N(r)) \right], \quad (\text{A.15})$$

where we have used Eq (2.5) and

$$\frac{d}{dr} \Pi(r, \omega) = -\frac{\omega^2}{2\kappa^2} (N(r) g_{rr} g^{tt} \phi(r) - \frac{1}{\omega^2} M(r) \phi(r)). \quad (\text{A.16})$$

If we define $\sigma(r, \omega) = i \Im(\sigma(r, \omega)) + \Re(\sigma(r, \omega))$, then we get

$$\frac{d}{dr} \Re(\sigma(r, \omega)) = -\omega \frac{4\kappa^2}{N(r)} \Re(\sigma(r, \omega)) \Im(\sigma(r, \omega)), \quad (\text{A.17})$$

$$\frac{d}{dr} \Im(\sigma(r, \omega)) = \omega \frac{2\kappa^2}{N(r)} \left[\left(\Re(\sigma(r, \omega)) \right)^2 - \left(\Im(\sigma(r, \omega)) \right)^2 + \left(\frac{1}{2\kappa^2} \right)^2 (N^2(r) g_{rr} g^{tt} - \frac{1}{\omega^2} M(r) N(r)) \right]. \quad (\text{A.18})$$

By solving above equations perturbatively in ω or numerically we can get electrical conductivity away from low frequency limit. Let us note that, if we take spatial momentum also to be non zero then gauge field fluctuation and metric fluctuation no longer decouple. In this case computation may become much more subtle.

B Condition on energy momentum tensor

Let us consider a constant r hypersurface outside the horizon. The unit normal vector to that hypersurface is $n^\mu \partial_\mu = n^r \partial_r$, where $n^r = \sqrt{g^{rr}}$. One can define the extrinsic curvature $\Theta_{\mu\nu}$ of the hypersurface to be

$$\Theta_{\mu\nu} = -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu). \quad (\text{B.1})$$

Using the form of the metric as in Eq (2.2), we get

$$\Theta_{tt} = -\frac{1}{2}\sqrt{g^{rr}}\frac{d}{dr}g_{tt} \quad , \quad \Theta_{xx} = -\frac{1}{2}\sqrt{g^{rr}}\frac{d}{dr}g_{xx}. \quad (\text{B.2})$$

Using Eq (3.6) and Eq (3.5), we can write

$$\sqrt{g}R_t^t = \frac{d}{dr}(\sqrt{h}\Theta_t^t), \quad \sqrt{g}R_x^x = \frac{d}{dr}(\sqrt{h}\Theta_x^x), \quad (\text{B.3})$$

where h is the determinant of the induced metric on the hypersurface. The induced metric on the constant r hypersurface is given by

$$\begin{aligned} ds_\Sigma^2 &= h_{tt}dt^2 + h_{xx}\sum_{i=1}^{d-1}(dx^i)^2 \\ &= g_{tt}dt^2 + g_{xx}\sum_{i=1}^{d-1}(dx^i)^2. \end{aligned} \quad (\text{B.4})$$

Let us define a tangent null vector $l^\mu\partial_\mu = \sqrt{-g^{tt}}\partial_t + \sqrt{g^{xx}}\partial_x$. Now we can write Eq (3.7) and consequently Eq (3.12) as

$$\begin{aligned} \sqrt{-g}R_{\mu\nu}l^\mu l^\nu &= \sqrt{-g}T_{\mu\nu}^{Total}l^\mu l^\nu \\ &= \sqrt{-g}T_{\mu\nu}^{E.M.}l^\mu l^\nu + \sqrt{-g}T_{\mu\nu}^{Matter}l^\mu l^\nu, \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \sqrt{-h}\Theta_{\mu\nu}l^\mu l^\nu \Big|_{r_h}^r &= \int_{r_h}^r dr \sqrt{-g}T_{\mu\nu}^{E.M.}l^\mu l^\nu + \int_{r_h}^r dr \sqrt{-g}T_{\mu\nu}^{Matter}l^\mu l^\nu \\ &= -\kappa^2 \rho A_t \Big|_{r_h}^r + \int_{r_h}^r dr \sqrt{-g}T_{\mu\nu}^{Matter}l^\mu l^\nu, \end{aligned} \quad (\text{B.6})$$

respectively. Upon using the Einstein equation (3.4) and the fact that for the metric of the form given in Eq (2.2), the R_{xt} component of the Ricci tensor is zero, we get $T_{tx}^{Matter} = 0$, since $T_{tx}^{E.M.} = 0$. So the condition that we get on the energy momentum tensor⁹ in Eq (3.13) can be written as

$$T_{\mu\nu}^{Matter}l^\mu l^\nu = 0. \quad (\text{B.7})$$

References

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⁹According to null energy condition, $T_{\mu\nu}^{total}l^\mu l^\nu \geq 0$, with l^μ a null vector. Since $T_{\mu\nu}^{E.M.}l^\mu l^\nu \geq 0$, the contribution from the matter part $T_{\mu\nu}^{Matter}l^\mu l^\nu$ may be negative as well. However it is interesting to note that, if we take a limit where charge of the black hole vanishes then $T_{\mu\nu}^{E.M.}l^\mu l^\nu = 0$, so that null energy condition gives $T_{\mu\nu}^{Matter}l^\mu l^\nu \geq 0$. So if we are interested in the backgrounds where matter sector does not act as source for electromagnetic field, it appears that $T_{\mu\nu}^{Matter}l^\mu l^\nu \geq 0$, irrespective of presence of gauge fields.

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